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DESCRIPTION OF MATHEMATICS. DESCRIPTION IN MATHEMATICS. MATHEMATICS AS A WAY OF DESCRIBING[1]

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Annual Lecture dedicated to the memory of Professor Andrzej Lasota

1. Description of mathematics

A. When asked about the nature of mathematics, probably the simplest (and, likely, most frequent) answer we would give is that mathematics is a *deductive* science. One can (or, perhaps, one should?) agree with that. But we also ought to notice, that the development of mathematics is purely *inductive.*

Whenever we consider a theory, with well-founded axioms and primitive notions, we never develop such a theory by listing all correct sentences within this theory and then verifying whether they are correct or not (that is, we never consider all correctly stated conjectures and check whether they are theorems). "Candidates for theorems" are picked according to an "informal" key (i.e. trying to generalize some already known theorems, capture analogies between results from other theories, seek answers to questions posed by physics, biology, social sciences etc. by first formulating appropriate mathematical models, and then – within such models – considering conceivable theorems that would provide answers to the original questions).

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[1]Translated from Polish to English by Paweł Gładki.

B. If we agree to the above, we inevitably face the "eternal" question about the nature of mathematics in its "philosophical dimension" – namely the question, whether this inductive development of mathematics relies more on *discovery* or *invention.* I touch on this question with much hesitation.

The question whether mathematicians *discover* or *invent* has been in consideration for a long time – "since forever". It seems that most mathematicians would favor the former, but it is hard to judge if that feeling has a quantitative basis.

The author of this text dares to state his own, highly subjective, opinions. He would like to avoid statements such as: "it is so", because, in fact, he usually does not know if "it is so". However, he does not avoid sharing with the reader his own feelings and quite often elaborates on what seems to him to be true. He counts on the reader's politeness to forgive him the lack of philosophical accuracy in numerous statements as well as the negligence in citing the vast literature on the subject, and – here deliberate – informality in mathematical considerations. It seems that the idea behind this exposition can be presented without much rigor.

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A genius sculptor (Michelangelo?) was supposed to say, while staring at a block of marble: *There is already a statue inside, I am only removing the extraneous material and it will come out on its own.*

Does it actually mean, that he *discovered* that the sculpture is there (that is, *discovered* the statue), or rather that he *discovered the possibility of "inventing"* (by "removing the superfluous material")?

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If instead of sculpting in marble, he would be using clay, so that rather than "removing the extraneous material" he would be "adding" it (or, at least, processing), would he also say that *the statue is already inside, but needs to be formed?* Probably not. He would probably say, that he is inventing the statue, but – other than by "removing the extraneous material" – by *inventing it "out of nothing".* The "nothing" here does not mean the clay, but the fact that at the beginning of the process he started from "ground zero", without *discovering* (that the statue is already there, and it remains to let it *come out* by "removing the extraneous material").

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I shall risk to propose taking the soundness of the following constatation under consideration: in both instances described above one can think of *discovering the possibility of inventing.*

* * *

In the extremely interesting book *Mathematics today: twelve informal essays* edited by L. A. Steen and Allen Hammond [4] in his essay quotes an excerpt from the discussion between Lipman Bers and Dennis Sullivan:

Lipman Bers: Do you invent or do you discover? What is your gut feeling?

Dennis Sullivan: Sometimes you come upon something sort of natural, it's like you're discovering it. But sometimes you just make something up out of thin air, so to speak; maybe force it a little bit.

Hammond himself presents his views in a broader context, along with referring, in a highly general way, the views of – in his opinion – most mathematicians (or even: the vast majority of mathematicians). Here are the excerpts from this text:

The argument among mathematicians over whether mathematical truths are invented or discovered has been going on a long time. It is not an argument that is easily resolved but it is revealing of how mathematicians think about their work. The two points of view are at first glance quite distinct. One holds that mathematicians discover a piece of reality [...], a truth not of their own making but rather an inherent part of the universe. "God made the integers", as one nineteenth century mathematician put it [...]. Hence the properties of the integers encompassed in simple arithmetic and in the more sophisticated theorems of number theory are viewed by most mathematicians in much the same way as astronomers think of the planets – discovered elements of the heavens. This absolutist or Platonist viewpoint – mathematics as reality revealed – extends to other areas of mathematics as well and is in fact the dominant dogma in the mathematical community. [...]

The second point of view emphasizes the role of human creativity in inventing mathematical structures. Clearly there is an element of human creativity involved, but how much, where to draw the line? Any system of mathematics rests ultimately on a series of axioms, for example, and there is in many instances an element of choice as to which axioms to use. Euclidean geometry was based on five supposedly basic and self-evident axioms about the nature of space, but just how arbitrary such choices can be was shown by the discovery (in physics, not in mathematics) that space is not Euclidean after all but rather Riemannian - that an alternate set of axioms due to Riemann provided a geometry that corresponds more to physical reality than that of Euclid's. Other mathematicians have since invented and explored the properties of still other geometries, all good mathematics, but with an element of choice. Nor is this element of human inventiveness confined to geometry; there exist alternate, "nonstandard" models of the real number system too. Even the laws of logic on which all of mathematical reasoning depends are not universally regarded as absolute. Some mathematicians have argued that there too there is an element of choice and convention. In fact many mathematicians will admit in *private that they think they create something, a good example or a good idea. In its most extreme form, this humanist or constructivist outlook rejects the idea of mathematicians as passive discoverers of a remote reality and instead asserts that mathematical phenomena are created by humans alone and do not otherwise exist.*

[...] A theorem may be discovered but its proof is usually invented. [...]

Einstein is reported to have felt that he invented the concept of relativistic spacetime, after the fact, but that he felt he was discovering an aspect of reality while he was doing the work. It is a feeling familiar to many mathematicians: "we are all Platonists in the trenches", as one put it. But the debate continues wherever mathematicians gather. "Do you invent or do you discover?"

Let us add, that the dispute among mathematicians and philosophers that has been on for centuries, and that is regarded with the issue of *discovering* vs. *inventing* mathematical entities[2], was recently boosted by a number of statements published by prominent mathematicians with the Newsletters of European Mathematical Society (see [2, 5, 6, 7]), whose differences in opinions (quite often far from being dramatic) can be already seen in the titles of their papers (compare, for example, [2] and [6]).

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Putting the crux of the matter aside, let us consider the language used to describe the (present) mathematics. One says (or, shall we say, one writes) quite universally[3] about the construction of the real numbers, or the construction of a basis of some space. While presenting proofs, one often says: *somebody gave a proof.* Is the thought (perhaps subconscious?) hiding behind these phraseological patterns that somebody actually *invented the proof,* or rather *found* (hence – *discovered*) the proof? Perhaps our feeling is that – that somebody – *found (discovered) a possibility of a proof* (that is – discovered the *possibility of inventing the proof*)?

Probably matters are quite similar when it comes to presenting examples or, so called, counter examples. One often says that somebody *found* (hence *discovered*) an example or a counter example, but also that somebody *invented* such an example or counter example. One also very often says that somebody *gave* an example (a counter example). In this last case one may, perhaps, admit both intuitive interpretations: *found* since one first *discovered*, or *found* since *invented.*

These remarks apply directly to the aforementioned words by Sullivan and

[2] This is the term I let myself use here, fully realizing it may be taken as an abuse of the widely accepted nomenclature, yet hoping that I shall succeed in avoiding confusion, and philosophers among the readers shall show enough understanding.

[3] Perhaps one should say *frequently* rather than *universally,* or use both terms simultaneously?

– according to the author's intentions – are supposed to somewhat widen their meaning and (*implicite,* at least) illustrate them.

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The aforementioned quotation: *The God made the real numbers,* is often given in a slightly different form: *The God made the integers, the whole rest is made by mathematicians.*

But perhaps: *The God made the formal logic... and the whole rest is made by mathematicians?*

Would it make (more) sense to (analogously) say: *The God made the musical scale (basic sounds), and the whole rest is made by composers?* Let us note, that here we speak of *composing* (hence *inventing*) of musical pieces (let us stress – *pieces*); hence in this context we clearly speak of what was *created* by composers.

It seems that a similar question can be asked with regard to works of art *created* by painters, or, in general, by all sorts of artists. Let us note, that the term *"creating"* itself suggests that we intuitively (but, really, just intuitively?) are convinced that here we are dealing (more) with *inventing* than *discovering.* Or is it perhaps that both in music and painting we *discover the possibility of invention* (composition!) from (already existing) sounds or colors?

Finally, is it that the *creation* in writing is in some sense similar to the creation in music or plastic arts? Can we assume that a poet, who wrote a poem, *discovered the possibility of inventing* a poem?

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I shall risk here a digression regarding theories specific for natural sciences. Before doing so I shall remark that I intend to remain on a – primitive one, in fact – intuitive ground, not even trying to consider whether, for example, there is a difference, or what sort of a difference is there between *the existence of a "mathematical entity"* and *the existence of a "material entity".* I would only like to share an observation (a suspicion?) regarding analogies between *the possibility of invention* (discovery of such a possibility?) in mathematics and, for example, in theoretical physics.

If we agree that a physicist *discovers* the structure of the matter, the question remains how does he describe it. Most often (I think) one speaks of this or that *theory*, which I – shall risk to – reduce to *the construction of a mathematical model (mathematical models).* Could we then say that a physicist *discovers the possibility of inventing a model?*

Quite often it happens so, that a model does not comply with our expectations (does not "reflect the reality sufficiently well"). One then either abandons such a model, or "improves" it, hence *de facto* "proposes" (*invents*) a new one. Is it not yet another *discovery of the possibility of inventing of a ("better") model?*

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Let us conclude these pictorial analogies with one more, perhaps a bit risky, attempt. Climbers often speak of leading first ascents of new routes in the Tatras or in the Alps. One may think that such a route is *discovered*, however those, who *lead* it may feel (or just feel?), that they *invented* it in some way. Perhaps it makes sense then to say that they *discovered the possibility of climbing the route,* hence *discovered the possibility of finding the route*, which, perhaps, can be compared to *discovering the possibility of inventing the route.*

One might be tempted to consider continuing this "metaphor-like" way of thinking by observing, that there exist alpine routes equipped in, other that the bare rock, "helpers"; when they were first introduced, the routes were scaled by means of aid climbing. Here *the construction* of the route is apparently literal. It should not be thought of as an abuse to compare – within the convention that we accepted – such *constructions* to considerations lead by mathematicians, who "go beyond" an already known theory, "adding" new apparatus to their concepts. As an example one may think of the theory of distributions, which arose from generalizing the theory of functions ("going beyond" this theory). Let us add that some problems (for example in differential equations) can not be solved by classical means on the grounds of the theory of functions. However, if we find their solutions by using the theory of distributions, we might "come back" to the theory of functions, as certain solutions that *a priori* are distributions sometimes turn out to be functions (to be more precise: distributions that can be identified with functions).

I feel this is a *discovery of the possibility of such a construction.* With regards to the aforementioned example – *the discovery of the possibility of building the theory of distributions* rather than *the discovery of distributions.*

* * *

One of widely proposed ways of teaching "higher mathematics" is the one that starts from the arithmetic of the integers, and then the reals. How are they "introduced"? We start either from the integers alone, or from the natural numbers first, and after providing their axioms we speak of *the construction of the integers* as a ring with the natural numbers (with addition and multiplication). As a next step, we – and this is how we actually call it! – *construct* the rationals, or the field of the rational numbers, to be more specific, where we "embed" through a certain isomorphism the ring of the integers[4].

[4] We usually proceed here with a standard reasoning, which is said to be the *construction* of the field of fractions of the ring of the integers. Such a reasoning (such a construction) is possible for every ring satisfying certain conditions, that are satisfied by the ring of the integers (to be *a commutative ring with identity and with no zero divisors*). We can, however, derive the rationals (*discover* or *construct*?) in more than just one way. In [3] we find the following sections: *3.3 The construction of the rationals by equivalence classes, 3.4 The construction of the rationals without the use of the natural numbers.* There is

When we have the rationals at our disposal, we *construct* (and, again, this is how we actually call it!) the reals. One can achieve this in more than one way. For example, one can introduce the Dedekind cuts or the equivalence relation, call it "≡", in the set of all sequences of rational numbers:

$$(')\{a_n\} \equiv \{b_n\} \overset{(def.)}{\Longleftrightarrow} \text{ the sequence } \{a_n - b_n\} \text{ converges to zero when } n \to \infty.$$

In the first case *the real numbers* are the *Dedekind cuts*, in the second one – *equivalence classes of sequences* with respect to the relation (′).

Did we, using one of the aforementioned ways, *discovered* the reals? And, if so, which ones? – the ones identified with the Dedekind cuts, or the ones "described in the other way"?

Could we rather say that, by presenting the two (or even three, if counting the method from the book [3] mentioned in the footnote 4) ways of obtaining the real numbers we *discovered the possibility of their construction (the construction of the field of real numbers)*, and even more than that, we did it in two (or even three) ways?

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Let us follow the thread sketched in what Hammond had said about different geometries and actually push it a bit further. Has Euclidean geometry been *discovered*? Have Riemannian or hyperbolic geometries been *discovered*? If one speaks of discovery here, one, perhaps, ought to refer to the fact that the Riemannian geometry provides a pretty good model for the universe, whereas the Euclidean model is not good enough. Does it mean, however, that the Euclidean geometry is "good for nothing" as a mathematical theory? I do not suppose anybody thinks that way. Regardless, if it *has been discovered, it can not be tossed away;* if *invented*, probably neither as well. It is, we have to admit, a beautiful mathematical theory, which *nota bene* provides nevertheless a good – or, in the context discussed, we shall say: practical, useful – model of the closest to us, macroscopic "physical reality".

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In an attempt to conclude I shall repeat what *explicite*, or – at least – *implicite* I have been trying to present in the form of verbalizing my feeling: IN MATHEMATICS WE DISCOVER THE POSSIBILITY OF INVENTING.

This should not be stated without the remark, that it is sometimes difficult to draw the line between what is "given" (that is, what is *discovered* or is *being discovered*), and what is *invented*. That is, perhaps, primarily a matter

3.5 The construction of the nonnegative reals and *3.6 The relative numbers (positive and negative)* where the reals are defined (*constructed*) in a way different from any of the two described here.

of a subjective feeling associated, among others, to the starting point of a conceivable construction.

Towards the end of these considerations, let me remark with, perhaps, the most risky statement (a confession about a subjective feeling, that the author finds the strongest one and well motivated by what has been presented above):

Perhaps it is the best to say, that... *there exist – in a platonic sense – not particular theories (such as the theory of integers, or reals, or different geometries etc.* so much, but THE MATHEMATICS AS A WHOLE, and we (within her limits) DISCOVER THE POSSIBILITY OF INVENTING various mathematical entities (objects), as well as mathematical theories.

If one agrees to this, then – in compliance with the "metaphor-like" remarks from the last paragraph of Section 4 – one may risk to say that mathematics "is given" as a rock face, and that when doing mathematics we *discover possibilities of inventing routes* (on that face). Let us add that, in fact, this face is mostly covered in clouds, which results in us seeing just its small fragments; consecutive *constructions* may lead us to see other fragments and may allow us (well, they usually allow) to *discover possibilities* of new *constructions (continuations of routes already constructed).*

However, this "metaphor-like" analogy must end here, as, unlike during even the longest of all alpine climbs, in mathematics we shall never reach "the end" – this "mathematical face" is not bounded. Nonetheless, we know (by Gödel's theorems) that for every mathematical theory rich enough[5] there exist (*will let us discover them* or *invent*?) statements formulated in the language of such a theory, that we can not determine, on the grounds of the theory under consideration, to be neither true nor false – they are *undecidable.* Hence adding such a statement to the theory under consideration as an additional axiom "expands" the theory significantly (here I would like to say that in this way we *discover the possibility of inventing* a new – "broader" – theory). Therefore every theory can be "expanded" in this way.

This unboundedness is truly fascinating and possibly one of the main motivation underlying the inspiration to do mathematics.

2. Description in mathematics (or – the language of mathematics)

While doing mathematics we use the rules of (binary) formal logic, and thus we also use the language of such logic and logical symbols when describing definitions, theorems and reasonings. One can say that we use a precise logical language. But one shall add straight away, that this formal language also

[5] *containing* – as we briefly say – *the arithmetic of natural numbers*

includes statements that not necessarily are written using logical symbols. To express the fact that the number g is the limit of the sequence $\{a_n\}$ we write – as we know it – in the logical symbolism as follows:

$$\forall \varepsilon > 0 \, \exists k \in \mathbb{N} \, \forall n \geq k \, (|a_n - g| < \varepsilon)^6,$$

or – equally precise! – in words as follows:

for every positive real number ε there exists a natural number k such that, for every natural number n no less than k, the following inequality is satisfied:

$$|a_n - g| < \varepsilon.$$

In the example the first statement is – in writing, in print – shorter than the second one. This is not always the case, especially when we present proofs. For example, the statement that is used quite frequently and actually makes a perfect mathematical sense, namely: without the loss of generality (usually, as we think of it – the generality of reasoning), written with the use of formal symbolic logic is lengthy (that is, takes a lot of space in printing) and may take a long time to read and interpret. That is all about – understood as above – the formal language of mathematics.

At the same time, along with the formal language we use a language that is almost common or almost literary[7] (that we shall simply call the "usual language"), avoiding an unnecessary use of formalisms and logical symbolism (or a symbolism introduced for the sake of the needs of a particular theory). We shall replace sequences of formal formulae with sequences expressed in the "usual language" (common, literary), on certain occasions making a reference to some analogies between simpler or better known situations, as well as using – at least *implicite* – some metaphors and abbreviations.

[6] We have skipped – except for one place, where we indicated that the number k belongs to the set of natural numbers – the ranges of variables appearing under quantifiers, assuming that – as we believe – one knows that ε is a real number, and n a natural number (which, actually, follows from the context). If, however, we would expect to see a fully formal precision in the definition here, we would have to add, that these variables belong – respectively – to the set of real numbers and to the set of natural numbers, just as we have done in the case of k "under the existential quantifier".

[7] The word "almost" was used twice not without a reason. Even though it happens so, that we use a common language in the most popular way of understanding the word "common" (first of all in elementary textbooks, or in popular articles), but quite often this is the "common language" in the understanding of a given group of mathematicians-specialists (in "advanced" texts, seminars etc.), that is, in fact, in the sense that perhaps would be better described by the term "a slang of a given theory", if it was not for rather pejorative connotations associated with it.

This interchanging between the fragments written in a formal language and fragments expressed using the “usual language” is, in fact, a rule in mathematical texts. However, proportions between the fragments of the former and the latter can vary substantially. One can consider these proportions in a ”qualitative” sense, as well as the “quantitative” one. That means, we shall be mostly interested in what is concerned with different ways of communicating between mathematicians (here the “quality” comes into play) and in how much of the text can be written formally vs. how much can be written in the “usual language” (the “qualitative” aspect).

Most importantly, we shall say when one can (or even should) limit, or perhaps even skip the use of a formalism, and when one is not allowed to do so. A lot (or even everything) depends on who is the addressee of a given mathematical text, that is who is the reader of a paper, or a textbook, or who is the listener of a lecture, or, finally, the participant of a seminar. If one can fear, that an “easy-going” text may be read improperly, one needs to decide on using a formalism. This relates mostly to papers presenting new results[8], as well as, to some extent, monographs. Textbooks should be written with the use of both forms of the just stated ways of transmitting information. What I mean here is the necessity to present new material with utmost precision and, at the same time, with making it more “accessible” by using less formal descriptions of certain objects, that is by, so called, geometric or physical “interpretations”, by exhibiting analogies between objects already known, or by employing the – already mentioned above – metaphors. Thus, for example, the description of the fact that the number g is the limit of the sequence $\{a_n\}$ can be less precisely, but, perhaps, more ”suggestively” presented as follows:

the number g is the limit of the sequence $\{a_n\}$ if[9] *its elements are located arbitrarily close to g, for n sufficiently large.*

This presentation can not appear **instead** of a precise definition in any elementary textbook in analysis (or calculus), but can greatly contribute to a formal way of lecturing.

I have mentioned the use of analogies and metaphors. It occurs, for example, when, while considering infinitely dimensional vector spaces, we lecture on their geometry, *de facto* referring to our intuitions associated with the (Euclidean) spaces of finite dimension, which, at the end, means imagining the discussed problems in the special case of a two-dimensional or three-dimensional space.

Also, whenever one is faced with exchanging strictly scientific information between specialists, one encounters such – seemingly – not very precise descriptions, which, however, do not lead to misunderstandings, and, in certain cases, shorten the text greatly. In such instances one can, perhaps, speak of

[8] First of all in these places where new objects are defined!

[9] In fact, one thinks here: *if and only if.*

a certain "quasi-coding" – of a precise mathematical text with a language far from being formal, or even a common language[10] (sometimes also with the use of "metaphorical" expressions) – by the "sender" of the information, followed by "decoding" by the "receiver". This is what happens quite often in seminar discussions.

The advantage of the "common" language over formalisms is clearly apparent when we speak of *mathematics* in popular texts of various levels of difficulty. Here I mean both texts presenting mathematics (or more often – its particular achievements and applications) to a broad audience of non-mathematicians, and also presenting – by specialists – certain branches of mathematics to mathematicians working in different areas. The latter takes place, for example, in the form of, so called, survey talks. Plenary lectures given by invited speakers at big meetings are usually of that character. In such lectures the use of informal language (which – let us remind one more time – does not have to mean the use of a common language in the popular understanding of this term) is almost a rule, and referring to analogies and metaphors is quite frequent.

Let us go back to the issue of the use of formal and "common" language, and let us consider it in a bit more exact fashion. Do we have – and, if so, when? – a guarantee that we will be able to flawlessly understand each other and carry on the information we want? When answering to this question, we must distinguish between the situation when the language is used by mathematicians among themselves, and the situation when it is used by mathematicians on one side and – shall we call them by default – "non-mathematicians" on the other one.

In the first case one can distinguish between two sub-cases. The first subcase is when we are considering a contact between specialists using the same language, that contains names (objects) specific to a given branch of mathematics (a mathematical theory) along with the whole formal apparatus. This is the case when there are no problems with transferring information. The formal language is precise, and fragments presented in the "common language" with conceivable use of analogies and metaphors are read flawlessly. If mathematicians of different branches try to communicate, certain difficulties can occur (and, usually, do occur) related to the specifics of different branches. These are, however, not so much linguistic difficulties (since the "common" language of mathematics is the same for all branches of mathematics, and specific terms and expressions are always formally well defined), but rather substantial. What has been written down can be properly read, but understanding what the text is actually trying to explain might be difficult without going deep into the theory that it regards.

[10] See footnote 3.

In the second case, when it comes to communication between mathematicians and non-mathematicians, the language of mathematics needs to be modified accordingly, or... replaced with a language from the areas bordering both the mathematics and the subject, whose representatives are the said "non-mathematicians", with whom the mathematicians need to communicate. This "frontier language" turns out to play a key role in numerous applications of mathematics.

Towards the end of this thread let us point to the sources of mathematical terminology. Every branch of mathematics has a unique, often quite specific, terminology. Basic branches already have a well-grounded and rich in tradition nomenclature, usually with well-correlated variants in different foreign languages. A newly introduced nomenclature, along with the development of particular theories, is primarily based on definitions – and these usually do exist – in English, quite often adapted to other languages without changes. The vast majority of names of objects appearing in mathematics (as well as names of structures and methods) has its roots in inspirations, that can be, without the risk of committing a serious omission, assumed to come from a certain sort of metaphors. We speak of *open sets, closed sets, boundary sets,* we define the *interior of a set, boundary of a set, condensation points of a set.* These names are well justified in the case of subsets of the set of real numbers, that is points on the real line, or points on the plane, or in the three-dimensional real space. The properties described by the aforementioned definitions have very natural, to the point of being intrusively obvious, geometric interpretations. Here, perhaps, one still can not see a deeper metaphor. But when we use these names in regard to general topological spaces, we no longer have such geometric interpretations at our disposal and we need to make a transition, a leap to a different situation ("a different reality"), that would correspond to a metaphor in the "common language".

As a side note to our considerations regarding the relations between formal and informal language, let me take the liberty to remark on what sort of an informal language is most frequently used in mathematical texts and, at the same time, risk an attempt to – at least partially – explain my reasons for making such a choice, not another one. It is well known that the language which is currently dominating in academical writings (at least regarding science and technology) is English, quite often called the Modern Latin. One can thus say, that it does not make any sense to comment on the domination of English in mathematical publications among others. I think, however, that it is worth noting that – regardless of how popular English is as an international language – mathematicians like to use English due to its certain convenient qualities, not always apparent in other languages. It is flexible as far as new terms are concerned. The rules of creating new names, which are supposed to contain phrases expressible, for example in Polish, by adjectives, allow to make compositions such as: noun-noun, or a name-noun, which plays a certain role, especially when it comes to lengthy and complicated texts, where it is important to catch up on all the key fragments in a possibly fastest way. For example, when we refer to *Banach fixed-point theorem,* the words *fixed-point* already focus on the most important aspect of the matter; we shall provide more examples such as this below. Finally, quite often English terms are shorter than the same terms in other languages, for example in French. Let us have a look at a few examples. *Banach space* corresponds to *le space de Banach* in French and *first order partial differential equations* corresponds to *équations différentielles aux derivées partielles du prémier ordre.*

A different issue is the choice of a language for popularizing mathematics. Here a good use of proper metaphors and references to analogies usually are very important. We have already remarked on this before.

3. Mathematics as a way of describing (that is: mathematics as a language of a description)

While naming this section, I thought of mathematics as a language that is used to the *world description.* Apparently, this particular "phrase" – as they say – exists in, so called, (almost) popular consciousness. I had been hesitating to instead call it "a clump of two phrases", but eventually have opted out, fearing possible pejorative connotations – but, perhaps, I should not have done so.

Is mathematics really (such a) language, and if so, is it **just** a language (of world description)?

It is possible to describe the world mathematically. The world is *mathematizable.* One sometimes says, *de facto* expressing perhaps the same – at least unintentionally – thought, that *the reality is describable mathematically.* Yet another version of the same, or at least similar, constatation (I would not like to try to go into detailed semantical analysis of conceivable differences, that could be subject to nontrivial logical and philosophical considerations, but seem negligible for what we are doing here), is stated as a theory, that *the nature of the world in mathematical,* or that *the world has a mathematical structure.*

If we agree to such a statement and decide that, really – using the shortest possible term, *the world is describable mathematically*, or even shorter – that *the world is mathematizable*, it would be difficult... not to be surprised! The question, why it is so, why, in fact, the *world is mathematizable* is a deep philosophical question, and for many also – at least partially – a theological one. I do not feel in a position to even try to answer this question. This is a subject for a separate dissertation. Such dissertations have been already written, by the way. Assuming that *the world is mathematizable,* let us think for a while what it really means. What does it mean, that mathematics is the language of its description? The shortest possible answer is: for physical phenomena (in the most broad sense of this term[11]) it is possible to find *mathematical models.* Due to natural limitations implied by the form of this presentation, I can not even try to precisely formulate what it means, in a broad sense, to construct models. I shall only stick to a few naive examples illustrating one particular – but very important – element of such constructions, namely what is commonly called "neglecting certain aspects of the phenomenon under consideration". If asked, how many kilometers of a telephone cable one needs to join two cities assuming a "straight line" that connects them, then the model would be – in fact – this straight line, a one-dimensional space, or more precisely: this space with a metric determined by the choice of a unit, for example one meter. Let us repeat – in this model we deal with a one-dimensional space, and that suffices. But if we asked, how many trucks one needs to carry this cable, we would have to compute its mass (weight). Weighting all of it would prove to be difficult, perhaps even impossible. We can, however, find this weight without using a scale. But we will need a three-dimensional model. We can, for example, find the volume of a cylinder whose base is a disc determined by a cross-section of a cable, and whose height is one meter, which would allow us to find its mass (that is the "weight of one meter of a cable", as we say), provided we know the density of the material used for manufacturing the cable, which would, in turn, allow us to find the mass of the whole cable[12]. These examples are trivial and – what one feels, requires an apology – rather naive. Still, they illustrate, especially the latter one, the most important issue in that matter. A mathematical model not only describes a situation, but also allows us to make some computations (and imply certain corollaries from these computations). This scheme of action is applied in much more complicated situations, quite often with the use of very subtle methods and advanced nomenclatural apparatus. And one obtains descriptions, that allow to conclude with surprisingly strong corollaries. I would wish to tell you about one more example from the stability theory. Numerous physical, chemical or biological processes are described by differential equations. We say that a certain state (fixed within the question under consideration) or a particular behaviour of the process is stable if small changes on the input data do not result in substantial differences in the output (or deviations from the state or its particular behaviour) in the time that is not bounded from above, and is asymptotically stable if: firstly – it is stable, and secondly – "the situation converges to" this fixed state or particular behaviour of the process.

[11] Which means, as seems to have been becoming quite apparent in the last years, for social phenomena as well.

[12] Here we silently assume for simplicity, that the cable is homogeneous, without insulation etc.

What was said above is, of course, far from being formally precise, yet the precision is not most important here; one would like to provide a "qualitative" outline of the problem, which is very important in numerous concrete applications. Namely, it deals with, to be as brief as possible, the question whether the process is "immune" to conceivable small errors made at the beginning in the sense that they do not result in dramatic changes ("errors") in the future. It turns out that whether we are guaranteed to achieve an asymptotical stability or not (that is the "immunity" against small errors) is decided by a simple algebraic condition, satisfied in the beginning by functions describing the process under consideration[13]. It is, indeed, intriguing that once we know that this condition is satisfied (which, for example, in the case of the processes described by a system of three differential equations is really easy to check) at the beginning, we already know about the important property of behaviour of solutions of these equations "up to infinity", where – shall we add – we do not need to know the formulae that describe these solutions. This certain algebraical condition checked at the beginning of the process determines it in the unbounded time, "up to infinity". I have devoted a lot of time to that particular example, since it provides – as I think – a good outlook on the power of mathematical methods, and, most importantly, it illustrates what we call the mathematical description of the world in the sense that it is not only a "passive" description, but also a research tool that usually allows us to obtain information about the *future* state of processes[14].

This is the sense in which I understand describing the world by mathematics. Describing through models where – obviously – one has to accept formal rules of mathematics.

Mathematics is thus *the language of world description*. In this context let us repeat once more the question from the beginning of this section: is it **just** a language?

I think that, if a *language* is only what we understand as a mathematical formalism[15], or – to say pictorially – just a "dictionary" and a "set of grammar rules", then the answer should be: **not**. Mathematics is **not only** a – so understood – language.

Let us start with an obvious statement. Literature or poetry, in any language, are, clearly, more than just a set of words and grammar rules.

[13] The following theorem is well-known to mathematicians: If the right hand side f of the system of differential equations (written in vector notation) $x' = f(x)$ is C^1, $f(0) = 0$, and the Jacobian of f at zero has characteristic roots that have negative real parts, the singleton $\{0\}$ is asymptotically stable.

[14] The theory of differential equations gives rise to theories with numerous direct applications. Such is the control theory. Thanks to this theory we are able to, among other things, travel into the space. Complicated mathematical models taken from this theory provide not only the *description*, but also allow us to *control* these space flight.

[15] Which, as we have already remarked, applies to the respectively chosen models

Just as, obviously, the literature (and even more so – the poetry) in any language shall not be reduced to a mere set of words and grammar rules (conceivably along with some rules or schemes of creating these or those works), mathematics shall not be reduced to a formal language of world description. It includes, nonetheless, elements that can not be considered forms of describing the world.

Can we consider a form of describing the world as a sentence properly formulated in a language of a given mathematical theory, which is undecidable within this theory (that is, one can not – within this theory – prove its correctness, nor show it is false)? And these statements are – shall we add – plentiful, which follows from the well-known (belonging to mathematics) *Gödel incompleteness theorem*[16]. Here we deal with an important "internal" property of the structure of mathematical theories, an imminent property of mathematics. One can thus, perhaps, say that mathematics is internally richer than a formal language of world description.

Moreover, it follows from Gödel theorem that every – rich enough – theory can be enriched by adding to its axioms a sentence that is (so far) undecidable. We have already discussed this.

A "purely practical observation" is as follows: for the – understood as above – world description one currently uses just a small part of what mathematics has to offer. One can say it differently, that just a small part of mathematics found so far applications in physics, chemistry, science in a broad sense, technology, or, finally, social sciences. Mathematics is being developed (one is tempted to say: develops itself) mostly independently of applications. Mathematicians are very happy whenever they find applications for the results they obtained, quite often they respond to requests from other sciences, but at the same time they do not give up developing mathematics for the sake of developing it itself, without looking at potential, or (already) visible applications. One can not risk to say that a theory, or a theorem would never find an application (so they would not serve the purpose of world description), but it can be said about a number of branches of mathematics, that they were built without the intent of finding any direct applications (and thus they can not be – right now – counted as components of the language of world description). Both in this sense and "practically" *mathematics turns out to be more than just the language of world description.*

I shall add one more remark. Mathematicians sometimes say that a theorem or a reasoning is *elegant*, or sometimes they even agree to say it is *beautiful.* Not being able, within the limits of this exposition, to develop this thread, I shall, in particular, skip the discussion on the subject on conditions that need to be satisfied in order for us to be able to elaborate on beauty in

[16] Kurt Gödel proved that in every theory "containing the theory of integers" (one can say less precisely, yet more pictorially: in every theory *rich enough*) one can formulate a sentence that can not be proven to be neither true nor false.

mathematics. I shall restrict myself and say that if it is a common feeling among those who make mathematics that it actually makes sense to speak of the beauty, beauty in mathematics, then it is perfectly justified to stress the – already expressed here – belief that mathematics is more than just a formal language of world description.

Since we have been talking about beauty here, one would expect to find attempts to elaborate on analogies between what we naturally associate with arts, in particular with music – as possibly the most "abstract" art. That might, however, lead the author of this exposition too far, and metaphors that would be conceivably used in these elaborations might cause the formal rigours and frames, apparent for a mathematician, to blow up. Thus I shall only repeat the constatation that is important in this context and ensure the reader, that one can really find elements of beauty in mathematics, and finding them is one of the most important motivations for those, for whom mathematics is a profession.

References

[1] Davies E.B., *Some recent articles about Platonism,* Newsletter of EMS **72** (2009), no. 6, 24–27.
[2] Davis P.J., *Why I am a (moderate) social Constructivist,* Newsletter of EMS **70** (2008), no. 12, 30–31.
[3] Grzegorczyk A., *Zarys arytmetyki teoretycznej (Outline of theoretical arithmetic)*, Biblioteka Matematyczna, vol. 39, Państwowe Wydawnictwo Naukowe, Warsaw, 1971 (in Polish).
[4] Hammond A.L., *Mathematics – our invisible culture*, in: *Mathematics Today: Twelve Informal Essays*, ed. by Steen L.A., Springer–Verlag, New York–Berlin, 1979, pp. 15–35.
[5] Hersh R., *On Platonism,* Newsletter of EMS **68** (2008), no. 6, 17–18.
[6] Mumford D., *Why I am a Platonist,* Newsletter of EMS **70** (2008), no. 12, 27–30.
[7] Mazur B., *Mathematical Platonism and its opposites,* Newsletter of EMS **68** (2008), no. 6, 19–21.

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