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Summary of the PhD dissertation

Existence of invariant densities for semiflows with jumps

In this dissertation we are going to analyze the problem of existence of invariant densities for piecewise deterministic Markow processes (PDMP) called semiflows with jumps. This class of processes is numerously applied in biological processes modeling. In particular, the existence of invariant densities is crucial for studying these models.

PDMP is a continuous–time process $\{X(t)\}_{t\geqslant 0}$ for which there exists an increasing sequence of so called jump times (t_n). Between two consecutive jumps the process is deterministic. We give a precise definition (based on [4]) of such semiflow in Chapter 1. We study PDMP using the theory of substochastic semigroups $\{P(t)\}_{t\geqslant 0}$ on L^1 space of functions integrable with respect to a fixed measure m. In Chapter 2 we define many necessary notions, such as substochastic semigroups and invariant densities. We also quote results which allow us to give sufficient conditions (based on [7]) for the existence and uniqueness of invariant densities for Markow operators. Moreover, we cite ([6]) theorems on asymptotic stability of substochastic semigroups.

In Chapter 3 we analyze the problem of existence of invariant densities for substochastic semigroups. Based on results from [9], we obtain the existence of a so called minimal substochastic semigroup $\{P(t)\}_{t\geqslant 0}$ for a process $\{X(t)\}_{t\geqslant 0}$ and for a unique Markow operator K on L^1 which satisfies: if the distribution of the random variable $X(0)$ has a density f, i.e.,

$$\Pr(X(0) \in B) = \int_B f(x)m(dx)$$

for all measurable subsets B of the space of states of the process, then $X(t_1)$ has a density Kf. Relationships between invariant densities for the operator K and invariant densities for the minimal semigroup $\{P(t)\}_{t\geqslant 0}$ are the main topic of this Chapter. Here the most important results are obtained in Theorems 3.5 and 3.12 and also in Corollary 3.14 following from these theorems. This part of dissertation is mainly based on [1].

In Chapter 4 we study the problem of existence of invariant densities for semiflows with jumps. One of the main result is Theorem 4.2 which gives sufficient conditions for the existence of a unique invariant density for a piecewise deterministic Markow process. As opposed to the article [3] and the monograph [5] we do not have to assume that the process is non-explosive and we look for absolutely continuous invariant measures. Additionally, we obtain asymptotic stability of the semigroup

$\{P(t)\}_{t\geqslant 0}$ (Theorem 4.6), i.e. the fact that the density of $X(t)$ converges to the invariant density in L^1 irrespective of the density of $X(0)$. In Section 4.2 we provide sufficient conditions for the existence of invariant densities and asymptotic stability of a semigroup $\{P(t)\}_{t\geqslant 0}$ in a form which makes the application easier, i.e., in terms of local characteristics of semiflows with jumps. In the final section of Chapter 4 we show how dynamical systems with random switching (as in [6]) can be studied with our methods. Also this Chapter is based on [1].

In the last two Chapters we present how to apply our results in analyzing biological models. These examples are a two dimensional model of gene expression with bursting (Chapter 5 based on [1]) and a fragmentation process (Chapter 6 which comes from [2]). Therefore, our framework can be used to analyze biological processes described by PDMPs ([8]), e.g. for gene expression with bursting, for dynamics with switching or for fragmentation process.

Literatura

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