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Summary of PhD Thesis

Extensions and ideals of associative rings

In the thesis all rings are associative but not necessarily commutative and with unity. Our aim is to determine the structure of associative rings using properties of their one-sided ideals.

We investigate how the transitivity of mixed versions of being one-sided ideal affects the structure of rings. Let x, y, z be elements from the set {left, right, twosided}. We are interested in the situation when an x-ideal of a y-ideal is a z-ideal of a ring. The class $\mathcal{K}(x, y; z)$ of rings in which such a relationship holds is called a Veldsman's class. These classes form a natural generalization of the class of filial rings and were introduced by Veldsman in [8].

In the context of the above problem it is important to know the structure and properties of extensions A of a given ring R in which R is a one-sided ideal. Such extensions were studied by many authors in various contexts. These considerations reduce to essential extensions of rings (i.e. such extensions A that every nonzero ideal of A has nonzero intersection with R). Therefore we focus our attention on them.

In the first chapter we present basic notions, definitions and facts used in the thesis. We introduce a definition of an (x, y)-subring of a ring. This definition turns out to be very useful tool for proving results of the fourth chapter. Moreover we give some new properties of simple domains.

In the second chapter we give necessary and sufficient conditions for existence of universal essential left ideal extensions of rings. This result is an analog of Flanigan's theorem about two-sided ideal extensions. K.I. Beidar in [2] introduced the notion of universal essential extension of a ring. Roughly speaking A is such an extension of R, if it is an essential ideal extension of R and every essential ideal extension of R can be embeded into A. He asked when a given ring has the universal ideal extension. It turned out that the answer to this question was already given by J.C. Flanigan in [4], but he used advanced methods and techniques of category theory. In [1] R.R. Andruszkiewicz presented a proof of Flanigan's theorem in the language of ring theory. In this part we also give a new elementary proof of the sufficient condition of Flanigan's theorem. Methods used in this proof allow us to answer the question raised by M. Petrich in [6] concerning pure extensions of rings.

In the third chapter we classify all classes of rings introduced in [3] which are generalizations of so-called $*$-simple rings (i.e. rings R which do not contain non-trivial ideals which are ideals in every ideal extension of R). These classes are related to radical theory. A $*(x, y; z)$-ideal of a ring R was defined in [3] as an x-ideal of R which is a z-ideal of any extension of R containing R as a y-ideal. Rings without non-trivial $*(x, y; z)$-ideals are called $*(x, y; z)$-simple. There are twenty seven such classes of rings. Descriptions of some of them were presented in [3]. We give the characterization of the remaining ones. In particular we prove that twelve of them are equal to the class of all rings and the description of fifteen remaining classes can be reduced to the description of four of them. Moreover we present a unified way of describing all those classes.

The fourth chapter is dedicated for issues related to Veldsman's classes $\mathcal{K}(x, y; z)$. Two among these classes, namely the classes of filial and left filial rings, were intensively studied in the literature. We investigate the structure of all twenty seven Veldsman's classes. It appears that one can select three of them which are fundamental in the description of remaining ones. The results obtained for those three classes together with known results on left filial rings allow us to determine the structure of rings from all remaining classes. We study relationships between the classes and get new their characterizations using the concept of (x, y)-subrings. We show that the prime radical of rings from the chosen three classes is a right duo ring which is T-nilpotent. Moreover it coincides with the sum of all nilpotent ideals. We also obtain structure theorems for algebras over a field, prime rings and domains from these classes. To do this we use some properties of selected radicals. Finally we prove that the class of reduced left filial rings is the largest left hereditary class contained in two of our three classes. Results presented in this chapter were published in [5].

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